Amplitudes and spinor-helicity in six dimensions

JHEP07(2009)075

(http://iopscience.iop.org/1126-6708/2009/07/075)

Published by IOP Publishing for SISSA

Received: *April 21, 2009*
Accepted: *June 26, 2009*
Published: *July 21, 2009*

Amplitudes and spinor-helicity in six dimensions

JHEP07(2009)075

Clifford Cheung and Donal O'Connell

Institute for Advanced Study, School of Natural Sciences,
Einstein Drive, Princeton, NJ 08540, U.S.A.

E-mail: cheung@ias.edu, donal@ias.edu

Abstract: The spinor-helicity formalism has become an invaluable tool for understanding the S-matrix of massless particles in four dimensions. In this paper we construct a spinor-helicity formalism in six dimensions, and apply it to derive compact expressions for the three, four and five point tree amplitudes of Yang-Mills theory. Using the KLT relations, it is a straightforward process to obtain amplitudes in linearized gravity from these Yang-Mills amplitudes; we demonstrate this by writing down the gravitational three and four point amplitudes. Because there is no conserved helicity in six dimensions, these amplitudes describe the scattering of all possible polarization states (as well as Kaluza-Klein excitations) in four dimensions upon dimensional reduction. We also briefly discuss a convenient formulation of the BCFW recursion relations in higher dimensions.

Keywords: Field Theories in Higher Dimensions, Gauge Symmetry, Global Symmetries

ArXiv ePrint: 0902.0981

doi:10.1088/1126-6708/2009/07/075

Contents

1 Introduction

The spinor-helicity formalism is the natural framework for representing on-shell scattering amplitudes of massless particles in four dimensions. This reflects a very basic result from field theory: asymptotic states of zero mass are uniquely specified by their momentum and helicity, and as such the S-matrix should be a function of these variables alone [1, 2]. Unfortunately, this structure is not manifest when amplitudes are represented using four-vectors and computed with conventional Feynman diagrams derived from a local action principle. In particular, for the case of gauge theory and gravity, the cost of manifest locality and Lorentz invariance is a gauge redundancy that must be introduced to eliminate extra propagating degrees of freedom. This gauge freedom implies that the external states are redundantly labeled by polarization vectors and that the amplitudes obey non-trivial Ward identities.

In contrast, the spinor-helicity formalism allows us to write down amplitudes without any mention of gauge symmetry or polarization vectors. Then, simple considerations of little group covariance of amplitudes are sufficient to strongly constrain or even determine the form of on-shell scattering amplitudes [3, 4]. From this point of view, the framework of spinor-helicity is not merely a computational trick, but is a way of representing amplitudes in their simplest, most physical form. Some very nice reviews of the four dimensional spinor-helicity formalism and its applications can be found in [5, 6].

Until now, there has not been a viable spinor-helicity formalism in more than four dimensions. There are, nonetheless, many reasons to suspect that a higher dimensional formalism should be both elegant and useful. In particular, many of the features of three and four dimensional spinors reflect their properties as representations of the $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ Lorentz groups. In six dimensions, the Lorentz group becomes $\mathrm{SL}(2, \mathbb{Q})$, where $\mathbb{Q}$ denotes the quaternions [7], so it seems probable that many of the features of the familiar four dimensional spinor-helicity variables have analogues in six dimensions.

In this paper, we construct a spinor helicity formalism in six dimensions. To orient the reader let us give a flavor of some of our results. The objects that we will consider are chiral and anti-chiral six dimensional spinors representing each external particle. For example, for particle 1, there is an associated chiral spinor $|1_a\rangle$, where the $a = 1, 2$ index transforms under one factor of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ little group of particle 1. The other $\mathrm{SU}(2)$ factor acts on the $\dot{a} = 1, 2$ index of an associated anti-chiral spinor $|1_{\dot{a}}]$. These little group indices will be ubiquitous in what follows, so it is worthwhile to comment on them briefly here. While these indices transform covariantly under the little group, we also know that they label the basis of physical states in the theory. As such, any free little group index will ultimately be contracted with some little group vector that labels the physical polarization of an external state. This is the point of view that we will adopt from here on.

Now, the momentum of particle 1 can be expressed as a product of either chiral spinors or anti-chiral spinors: $-4p_1^\mu = \langle 1^a|\sigma^\mu|1_a\rangle = [1^{\dot{a}}|\tilde{\sigma}^\mu|1_{\dot{a}}]$, where σ and $\tilde{\sigma}$ are the six dimensional Pauli matrices. These expressions for the momentum in six dimensions contrast with the four dimensional expression; in that case, momenta are given by product of one chiral and one anti-chiral spinor. With the spinors corresponding to particle 2, $|2_b\rangle$ and $|2_{\dot{b}}]$, we can construct a natural Lorentz invariant object, $\langle 1_a|2_{\dot{b}}]$, that connects the two particles. The advantages of this formalism are illustrated by the striking simplicity of on-shell scattering amplitudes, which we have computed up to five points. For example, as we will show, the color-ordered Yang-Mills four point amplitude is given by

$$A_4(1, 2, 3, 4) = -\frac{i}{st}\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}], \tag{1.1}$$

in terms of appropriate quadrilinear contractions of the spinors associated with each leg. We shall define this contraction in more detail below. Meanwhile the gravitational four point function is given by

$$\mathcal{M}_4(1,2,3,4) = \frac{i}{stu}\langle 1_a 2_b 3_c 4_d\rangle\langle 1_{a'} 2_{b'} 3_{c'} 4_{d'}\rangle[1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}][1_{\dot{a}'} 2_{\dot{b}'} 3_{\dot{c}'} 4_{\dot{d}'}]. \tag{1.2}$$

The amplitude for scattering of a general state, described by some appropriate little group tensor, is found by contracting the free indices of these expressions against the little group tensor. Recent work on the D-dimensional unitarity method [8] has some overlap with our results but the focus of our article is very different.

Upon dimensional reduction to four dimensions, we reproduce the usual expressions for gauge boson and graviton scattering in four dimensions. Furthermore, we obtain some four dimensional amplitudes for scalars: for example, amplitudes that describe scattering of longitudinal modes of KK vector bosons. From the gravitational amplitude we can obtain expression for gravitons scattering with gauge bosons, massive vector bosons and so on.

The structure of the paper is as follows. In section 2 we present a brief review of the spinor-helicity formalism in four dimensions. We then go on to develop the six dimensional framework in section 3. We use this formalism in section 4 to compute beautifully simple forms for the three point amplitudes in Yang-Mills theory and gravity. The unique kinematics at three points will require some new ingredients to express this answer. Section 5 contains some remarks on the BCFW recursion relations [9, 10] and a method for their efficient use in six dimensions. With this tool we derive the four point amplitudes in section 6 and the five point Yang-Mills amplitude in section 7, before concluding. The appendices contain some useful identities for manipulating six dimensional spinors.

2 A review of spinor-helicity in four dimensions

To begin, let us briefly review the spinor-helicity formalism in four dimensions. Much of this discussion will have a direct analogy in six dimensions. The basic point of spinor-helicity is to represent a light-like four-momentum p_μ as a bi-spinor

$$p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{2.1}$$

where λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are complex valued spinors transforming in the $(2,0)$ and $(0,2)$ representations of the Lorentz group. Since $p_\mu p^\mu = \det(p_{\alpha\dot{\alpha}})$ and $p_{\alpha\dot{\alpha}}$ is a rank one matrix, this bi-spinor represents a null four-vector. In order to fix p_μ to be real, we need to impose a reality condition, $\tilde{\lambda} = \pm\lambda^*$. However, it is often useful to analytically continue to complex momenta, so we frequently relax this condition. The recursion relations [9–15], which exploit the pole structure in complex momentum space to recursively relate higher point on-shell amplitudes to lower-point ones, are a specific instance of this.

While we have specified the momenta in terms of spinors, we know that a massless particle in four dimensions is labeled not just by its four-momentum p_μ, but also by its helicity $h = \pm$. Indeed, in D dimensions it is known that any massless particle is defined by a ket in a Hilbert space, $|p_\mu, h\rangle$, where h is a general label for a linear representation of the $\mathrm{SO}(D-2)$ little group, the subgroup of the Lorentz group that leaves p_μ invariant.

Under Lorentz transformations, the kets transform according to

$$|p_\mu, h\rangle \to \sum_{h'} W_{hh'} |\Lambda_\mu{}^\nu p_\nu, h'\rangle \tag{2.2}$$

where $\Lambda_\mu{}^\nu$ and $W_{hh'}$ are Lorentz and little group transformations, respectively. In four dimensions, the little group is SO(2), and so h simply labels the helicity; then $W_{hh'}$ is a diagonal matrix. For real four dimensional momenta, $p_{\alpha\dot{\alpha}}$ is manifestly invariant under

$$\lambda \to z\lambda, \quad \tilde{\lambda} \to z^{-1}\tilde{\lambda}, \tag{2.3}$$

where z is a phase for real momenta, or any non-zero complex number for complex momenta. This is the little group action on the spinor. With multiple external particles labeled by i, each spinor transforms under its own little group, so $\lambda_i \to z_i\lambda_i$. From general considerations [3] one can show that given helicity assignments $h_i = \pm$ and spins $s_i = 0, 1, 2$, an on-shell amplitude transforms as $\mathcal{M} \to \prod_i z_i^{2s_ih_i}\mathcal{M}$, which highly constrains the form of amplitudes.

Now that we understand the transformation properties of the spinors under the Lorentz and little groups, let us comment on Lorentz invariant products. Given two chiral spinors, $\lambda_{i\alpha}$ and $\lambda_{j\beta}$, there is an obvious Lorentz invariant product, $\lambda_{i\alpha}\lambda_{j\beta}\epsilon^{\alpha\beta} \equiv \langle\lambda_i\lambda_j\rangle \equiv \langle ij\rangle$, and likewise for two anti-chiral spinors, $\tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_{j\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} \equiv [\tilde{\lambda}_i\tilde{\lambda}_j] \equiv [ij]$. These objects are little group covariant since $\langle ij\rangle \to z_iz_j\langle ij\rangle$ and $[ij] \to z_i^{-1}z_j^{-1}[ij]$. All on-shell amplitudes are functions of these Lorentz invariant, little group covariant objects. For example, the three point function of a theory of spin s particles is

$$A_3(1^-, 2^-, 3^+) = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^s, \qquad A_3(1^+, 2^+, 3^-) = \left(\frac{[12]^3}{[23][31]}\right)^s, \tag{2.4}$$

$$A_3(1^-, 2^-, 3^-) = \frac{1}{M^2}\left(\langle 12\rangle\langle 23\rangle\langle 31\rangle\right)^s, \qquad A_3(1^+, 2^+, 3^+) = \frac{1}{M^2}\left([12][23][31]\right)^s \tag{2.5}$$

with no reference to polarization vectors. In eq. (2.5) we have included factors of $1/M^2$ on dimensional grounds; these amplitudes vanish in the case of pure Yang-Mills theories but arise from a dimension six operator $\operatorname{tr} F_{\mu\nu}F_\rho{}^\nu F^{\mu\rho}$ in an effective theory. In fact, eq. (2.4) and eq. (2.5) are the form of the three point amplitude to all orders in perturbation theory, as a consequence of momentum conservation and little group covariance [3, 4]. Beginning with this three point amplitude, the BCFW recursion relations can then be used to construct all higher point functions from these amplitudes.

That said, if we wish to make a direct connection to more conventional methods for computing amplitudes, then we can still define polarization vectors in terms of four dimensional spinors. Consider a particle of momentum p; it is convenient to denote the associated spinors as $\lambda = |p\rangle$ and $\tilde{\lambda} = |p]$. Then the polarization vectors associated with this particle can be written as

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}\frac{\langle p|\sigma^\mu|q]}{[pq]} \tag{2.6}$$

$$\varepsilon_+^\mu = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle pq\rangle}, \tag{2.7}$$

JHEP07(2009)075

where $|q\rangle$ and $|q]$ are reference spinors. Note that the polarization vectors are appropriately covariant under the little group of $|p\rangle$ and $|p]$, but are manifestly invariant under little group transformations acting on the reference spinors.

3 Constructing spinor-helicity in six dimensions

It is straightforward to extend the construction of the previous section to six dimensions. Our goal is to construct a spinor representation of the momentum, p_μ, that transforms appropriately under the Lorentz and little groups. In particular, since the Lorentz group is $SO(6) \simeq SU(4)$, these six dimensional spinors are complex four component objects, transforming in the fundamental of SU(4) under Lorentz transformations. Since the antisymmetric representation of SU(4) is six dimensional, we expect p_μ to be written as some antisymmetric product of two spinors. Moreover, since the little group is $SO(4) \simeq SU(2) \times SU(2)$ for real momenta, then the spinors should have two SU(2) spinor indices. For the purposes of this paper we consider complex momenta, for which the associated spinors need not satisfy any reality conditions. Consequently, the little group is extended to $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

3.1 From the Dirac equation

Indeed, solutions of the Dirac equation for a null momentum, p_μ, have the properties we require, as we will now see. The equations we must solve are

$$p_\mu \sigma^\mu{}_{AB} \lambda^B = 0, \quad p_\mu \tilde{\sigma}^{\mu AB} \tilde{\lambda}_B = 0, \tag{3.1}$$

where $\sigma^\mu{}_{AB}$ and $\tilde{\sigma}^{\mu AB}$ are six dimensional antisymmetric Pauli matrices described in detail in appendix A. Our choice of basis is such that the σ matrices restricted to $\mu = 0, 1, 2, 3$ reduce to a familiar (Weyl) choice of γ matrices in four dimensions:

$$\sigma^\mu = \begin{pmatrix} 0 & {}^{(4)}\sigma^{\mu\alpha}{}_{\dot{\alpha}} \\ -{}^{(4)}\sigma^{\mu\alpha}{}_{\dot{\alpha}}{}^T & 0 \end{pmatrix}, \quad \tilde{\sigma}^\mu = \begin{pmatrix} 0 & {}^{(4)}\sigma_{\mu\alpha}{}^{\dot{\alpha}} \\ -{}^{(4)}\sigma_{\mu\alpha}{}^{\dot{\alpha}T} & 0 \end{pmatrix}, \quad \mu = 0, 1, 2, 3, \tag{3.2}$$

where ${}^{(4)}\sigma^\mu{}_{\alpha\dot{\alpha}} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ are the usual four dimensional sigma matrices, and α, $\dot{\alpha} = 1, 2$ are spinor indices of the four dimensional Lorentz group. We take λ^A and $\tilde{\lambda}_A$, the solutions of eq. (3.1), to be in the fundamental and anti-fundamental representations of SU(4), respectively. Unlike the familiar case of SU(2), the fundamental and anti-fundamental representations of SU(4) are inequivalent since there is no tensor that can raise or lower indices. In fact, the only non-trivial invariant tensor is a four index object, ϵ_{ABCD}.

Since $p_\mu \sigma^\mu{}_{AB}$ is a rank two matrix, there is a two dimensional space of solutions for the λ equation in Equations (3.1) that we can label by $a = 1, 2$. We do the same for the $\tilde{\lambda}$ equation, labeling by $\dot{a} = 1, 2$. Thus, the chiral and anti-chiral spinors can be written as λ^{Aa} and $\tilde{\lambda}_{A\dot{a}}$.[1] We will see that these a and $\dot{a}$ indices are precisely the $SU(2) \times SU(2)$ indices of the little group.

[1] Including the little group label, each chiral and anti-chiral spinor is a four by two matrix. Consequently, we can reinterpret these objects as quaternionic two-component spinors, where each quaternion is represented by a two by two matrix. This reflects a fact we alluded to earlier: the Lorentz group in six dimensions is isomorphic to $SL(2, \mathbb{Q})$ [7].

JHEP07(2009)075

If the momentum p happens to lie in the privileged four-space fixed by our choice of the σ matrices, that is $p = (p^0, p^1, p^2, p^3, 0, 0)$, then we can choose solutions of eq. (3.1) given by

$$\lambda^A{}_a = \begin{pmatrix} 0 & {}^{(4)}\lambda_\alpha \\ {}^{(4)}\tilde{\lambda}^{\dot\alpha} & 0 \end{pmatrix}, \quad \tilde{\lambda}_{A\dot a} = \begin{pmatrix} 0 & {}^{(4)}\lambda^\alpha \\ {}^{(4)}\tilde{\lambda}_{\dot\alpha} & 0 \end{pmatrix}, \tag{3.3}$$

where ${}^{(4)}\lambda$ and ${}^{(4)}\tilde{\lambda}$ are four dimensional spinors. Note the position of the four dimensional spinor indices; these follow from the positions of the indices in eq. (3.2).

In computations, it is frequently convenient to use a bra-ket notation, and so we write

$$\lambda^a = |p^a\rangle, \quad \tilde{\lambda}_{\dot a} = |p_{\dot a}]. \tag{3.4}$$

When several particles scatter we will choose to label the kets by the label of the particle for brevity. It is possible to normalize the basis of spinor solutions so that

$$p_\mu \tilde{\sigma}^{\mu AB} = p^{AB} = \lambda^{Aa}\lambda^{Bb}\epsilon_{ab} = |p^a\rangle\epsilon_{ab}\langle p^b|, \tag{3.5}$$

$$p_\mu \sigma^\mu{}_{AB} = p_{AB} = \tilde{\lambda}_{A\dot a}\tilde{\lambda}_{B\dot b}\epsilon^{\dot a\dot b} = |p_{\dot a}]\epsilon^{\dot a\dot b}[p_{\dot b}|. \tag{3.6}$$

With the help of eq. (A.4), we can express the momentum vector itself in terms of the spinors as

$$p_\mu = -\frac{1}{4}\langle p^a|\sigma_\mu|p^b\rangle\epsilon_{ab} = -\frac{1}{4}[p_{\dot a}|\tilde{\sigma}_\mu|p_{\dot b}]\epsilon^{\dot a\dot b}. \tag{3.7}$$

From this point on, we will freely raise and lower the SU(2) indices, a and $\dot a$, using the definitions

$$|p_a\rangle = \epsilon_{ab}|p^b\rangle, \tag{3.8a}$$

$$|p^{\dot a}] = \epsilon^{\dot a\dot b}|p_{\dot b}]. \tag{3.8b}$$

We define $\epsilon^{12} = 1$ and $\epsilon_{12} = -1$.

3.2 To the little group

Earlier, we remarked that the a index of the spinor $|p_a\rangle$ transforms under the little group. Let us now take a moment to explain why this is so. Consider a Lorentz transformation Λ with the property that $\Lambda^\mu{}_\nu p^\nu = p^\mu$; that is, p is invariant under the transformation. Then Λ is an element of the SO(4) little group associated with p. This transformation acts on the spinor λ by a unitary matrix U and upon $\tilde{\lambda}$ by the inverse matrix U^{-1}. If we define $\lambda' = U\lambda$, then λ' satisfies the Dirac equation since

$$p_\mu\sigma^\mu\lambda' = p_\mu(U^{-1}U\sigma^\mu U\lambda) = \Lambda_\nu{}^\mu p_\mu(U^{-1}\sigma^\nu\lambda) = U^{-1}(p_\mu\sigma^\mu\lambda) = 0. \tag{3.9}$$

Consequently, we may write $\lambda'_a = M_a{}^b\lambda_b$ for some matrix M, as the spinors λ_a form a basis for the solution space. Using the two expressions for λ' it is straightforward to show that

$$-\frac{1}{4}\lambda'^a\sigma^\mu\lambda'_a = p^\mu = p^\mu \det M \tag{3.10}$$

Therefore we conclude that $M \in \mathrm{SL}(2,\mathbb{C})$. Similarly, the spinors $\tilde{\lambda}$ transform as $\tilde{\lambda}' = \tilde{M}\tilde{\lambda}$ where $\tilde{M} \in \mathrm{SL}(2,\mathbb{C})$. Since there is in general no relation between M and $\tilde{M}$ we conclude that the full space of transformations that leave the momentum invariant, i.e. the little group, is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$.

3.3 Invariants and covariants

In analogy with the four dimensional case we can now construct a set of natural Lorentz invariant, little group covariant objects.[2] Lorentz invariant contractions of spinors associated to two particles labeled by i and j are

$$\langle i^a | j_{\dot b}] = \lambda_i^{Aa} \tilde{\lambda}_{jA\dot b} = [j_{\dot b} | i^a\rangle, \tag{3.11}$$

which is a two by two matrix that transforms in the bifundamental under a separate SU(2) little group factor for particle i and for particle j. For spinors associated with momenta p and q in the privileged four-space of our σ matrices, we find

$$\langle i_a | j_{\dot b}] = \begin{pmatrix} -[ij] & 0 \\ 0 & \langle ij\rangle \end{pmatrix}_{a\dot b}, \tag{3.12}$$

$$[i_{\dot a} | j_b\rangle = \begin{pmatrix} [ij] & 0 \\ 0 & -\langle ij\rangle \end{pmatrix}_{\dot a b}. \tag{3.13}$$

Note that $\det[i|j\rangle = -2p_i \cdot p_j$. In addition, using the four index antisymmetric tensor, we can construct a Lorentz invariant from four spinors labeled by i, j, k, l:

$$\langle i^a j^b k^c l^d\rangle = \epsilon_{ABCD} \lambda_i^{Aa} \lambda_j^{Bb} \lambda_k^{Cc} \lambda_l^{Dd} \tag{3.14}$$

$$\left[i_{\dot a} j_{\dot b} k_{\dot c} l_{\dot d} \right] = \epsilon^{ABCD} \tilde{\lambda}_{iA\dot a} \tilde{\lambda}_{jB\dot b} \tilde{\lambda}_{kC\dot c} \tilde{\lambda}_{lD\dot d}. \tag{3.15}$$

Finally, given particles labeled by i, j, and $k_1, \ldots, k_{2n+1}$, we define

$$\langle i_a | \not{p}_{k_1} \not{p}_{k_2} \cdots \not{p}_{k_{2n+1}} | j_b\rangle = \lambda_i^{A_1}{}_a (p_{k_1} \cdot \sigma_{A_1A_2})(p_{k_2} \cdot \tilde{\sigma}^{A_2A_3}) \cdots (p_{k_{2n+1}} \cdot \sigma_{A_{2n+1}A_{2n+2}}) \lambda_m^{A_{2n+2}}{}_b \tag{3.16}$$

$$\langle i_a | \not{p}_{k_1} \not{p}_{k_2} \cdots \not{p}_{k_{2n}} | j_{\dot b}] = \lambda_i^{A_1}{}_a (p_{k_1} \cdot \sigma_{A_1A_2})(p_{k_2} \cdot \tilde{\sigma}^{A_2A_3}) \cdots (p_{k_{2n+1}} \cdot \tilde{\sigma}^{A_{2n}A_{2n+1}}) \tilde{\lambda}_{jA_{2n+1}\dot b}. \tag{3.17}$$

3.4 Polarization vectors

The advantage of the spinor-helicity formalism is that the polarization states of the external particles live in irreducible representations of the little group. In contrast, conventional Feynman diagrammatics forces us to represent polarization states redundantly as Lorentz six-vectors. In this section we make contact with this picture by writing polarization six-vectors in terms of the six dimensional spinors.

[2]The language here may seem a bit odd, because the little group is by definition the subgroup of the Lorentz group that leaves p_μ invariant. Thus we should expect that anything Lorentz invariant is little group invariant as well. While this is certainly true, the little group can also be understood as a separate set of transformations that acts on and defines the basis of polarizations for *each* external particle. For example, for a massive particle in four dimensions, the little group is SO(3)—thus, while an SO(3) little group index is of course rotated via boosts, it can also be thought of as an index that is to be contracted with some three-vector polarization built out of the basis polarizations. Throughout this paper, our view is that these indices label these physical polarizations of external states. Thus, when we say some object is Lorentz invariant but little group covariant, we mean a genuine Lorentz invariant, which happens to depend on the polarization states of the various particles scattering.

To begin, we pick a reference six-vector q such that $p \cdot q \neq 0$, where p is the particle momentum. Associated with q are two spinors such that $q = |q^a\rangle\langle q^b|\epsilon_{ab}$ and $q = |q_{\dot a}][q_{\dot b}|\epsilon^{\dot a\dot b}$. We then define the polarization vectors to be

$$\varepsilon^{\mu}{}_{a\dot a} = \frac{1}{\sqrt{2}} \langle p_a|\sigma^\mu|q_b\rangle \left(\langle q_b|p^{\dot a}]\right)^{-1} \tag{3.18}$$

$$= \frac{1}{\sqrt{2}} \left(\langle p^a|q_{\dot b}]\right)^{-1} [q_{\dot b}|\tilde\sigma^\mu|p_{\dot a}]. \tag{3.19}$$

We note that, in contrast to the four dimensional case, the polarizations are not simply labeled by helicity + or -, but by $SO(4) \simeq SU(2)\times SU(2)$ little group indices. On the other hand, just as in four dimensions, a little group transformation acting on the reference spinors $|q\rangle$ and $|q]$ has no effect on the polarization. We have normalized the polarization vectors so that

$$\varepsilon^{\mu}{}_{a\dot a}\varepsilon_{\mu b\dot b} = \epsilon_{ab}\epsilon_{\dot a\dot b}. \tag{3.20}$$

On physical grounds, the polarization vectors must satisfy two key properties: they must transform appropriately under gauge transformations, and furthermore they must form a complete set of vectors transverse to the momentum p. Let us demonstrate that our vectors satisfy these requirements, starting with the former. Choose a new gauge q' such that $p\cdot q' \neq 0$; associated with this new gauge are new spinors $|q'\rangle$. In general, we can write

$$|q'_c\rangle = A_c{}^b|q_b\rangle + B_c{}^a|p_a\rangle, \tag{3.21}$$

where $[p_{\dot a}|q'_c\rangle = A_c{}^b\langle q_b|p_{\dot a}]$. Now, since $\det[p|q\rangle = -2p\cdot q \neq 0$ and similarly $\det[p|q'\rangle \neq 0$, it follows that $\det A \neq 0$ so that A is an invertible matrix. Using the definition of the polarization vectors, it is now a straightforward calculation to show that

$$\varepsilon'^{\mu}{}_{a\dot a} = \varepsilon^{\mu}{}_{a\dot a} + \Omega_{a\dot a}p^\mu, \tag{3.22}$$

where

$$\Omega_{a\dot a} = -\sqrt{2}(A_c{}^b\langle q_b|p^{\dot a}]\,)^{-1}B_{ca}. \tag{3.23}$$

Thus, the polarization vectors shift under a gauge transformation by an amount proportional to the associated momentum, as desired. Finally, it is a straightforward computation to show that the polarization vectors form a complete set in the sense that

$$\varepsilon^{\mu}{}_{a\dot a}\varepsilon^{\nu a\dot a} = \eta^{\mu\nu} - \frac{1}{p\cdot q}(p^\mu q^\nu + p^\nu q^\mu). \tag{3.24}$$

4 The three point function

In this section, we derive compact forms for the three point scattering amplitudes of Yang-Mills theory and gravity. It will actually be illuminating to first try and guess the form of the three point amplitude directly from little group considerations alone. In particular, given particles 1, 2, and 3, with little group indices $(a, \dot a)$, $(b, \dot b)$, and $(c, \dot c)$, we know that the amplitude must have exactly one of each index. The most obvious guess is $\langle 1_a|2_{\dot b}]\langle 2_b|3_{\dot c}]\langle 3_c|1_{\dot a}]/M^2$, where some scale M has been included on dimensional grounds. As it turns out, this amplitude arises precisely from the higher dimension operator $\mathrm{tr}\, F_{\mu\nu}F^{\nu\rho}F_\rho{}^\mu/M^2$ which can be added to Yang-Mills theory. The four dimensional analogs of this amplitude are $A_3(1^-2^-3^-)$ and $A_3(1^+2^+3^+)$. If we are concerned with the renormalizable couplings of Yang-Mills theory, then no such scale M is present, and moreover momentum conservation forces all the kinematic invariants $p_i \cdot p_j$ to vanish. Thus, dimensional analysis tells us that to write down the three point amplitude for Yang-Mills theory, it will be necessary to invert the quantities $\langle i|j]$. However, this is naively a problem because $\det\langle i|j] = -2p_i \cdot p_j = 0$. In this way, we see that a new ingredient is necessary.

The solution to this problem is as follows. Since $\langle i_a|j_{\dot b}]$ is a rank one matrix, it can be expressed as a product of two two-component objects, u_{ia} and $\tilde u_{j\dot b}$, such that $\langle i_a|j_{\dot b}] = u_{ia}\tilde u_{j\dot b}$. Since u_i and $\tilde u_j$ are quite reminiscent of four dimensional spinors, we know the natural inversion of these u's. In particular, we introduce spinors w_i and $\tilde w_j$ such that $u_i^a w_{ia} = 1$ and $\tilde u_j^a \tilde w_{ja} = 1$. Of course, these inverses are not uniquely defined, and we will discuss this ambiguity in detail below. However, we ultimately find a factorized form for the Yang-Mills three point function given by

$$A_3(1_{a\dot a}, 2_{b\dot b}, 3_{c\dot c}) = i\Gamma_{abc}\tilde\Gamma_{\dot a\dot b\dot c}, \tag{4.1}$$

where the tensors Γ and $\tilde\Gamma$ are simply

$$\Gamma_{abc} = u_{1a}u_{2b}w_{3c} + u_{1a}w_{2b}u_{3c} + w_{1a}u_{2b}u_{3c}, \tag{4.2}$$

$$\tilde\Gamma_{\dot a\dot b\dot c} = \tilde u_{1\dot a}\tilde u_{2\dot b}\tilde w_{3\dot c} + \tilde u_{1\dot a}\tilde w_{2\dot b}\tilde u_{3\dot c} + \tilde w_{1\dot a}\tilde u_{2\dot b}\tilde u_{3\dot c}. \tag{4.3}$$

We will frequently use notation like $A_n(1_{a\dot a}, 2_{b\dot b}, \ldots)$ to indicate an n point gauge theory amplitude where the little group indices of particle one are $(a, \dot a)$ and so on.

4.1 Three point amplitude in Yang-Mills

Let us now discuss these issues in detail. It is helpful for the purposes of clarity to choose p_1, p_2 and p_3 in the privileged four-space of our choice of the σ matrices. Then the Lorentz invariant brackets can be taken to be of the form

$$\langle 1_a|2_{\dot b}] = \begin{pmatrix} -[12] & 0 \\ 0 & \langle 12\rangle \end{pmatrix}. \tag{4.4}$$

From our experience with the three point function in four dimensions, we know that either $\langle 12\rangle = 0$ or $[12] = 0$. We suppose that $[12] = 0$. Thus, the Lorentz invariants are of the form

$$\langle 1_a|2_{\dot b}] = \begin{pmatrix} 0 & 0 \\ 0 & \langle 12\rangle \end{pmatrix}. \tag{4.5}$$

Now we can define two component vectors u_i and $\tilde u_i$ for $i = 1, 2, 3$ such that the equations

$$\langle 1_a|2_{\dot b}] = u_{1a}\tilde u_{2\dot b} \qquad \langle 1_a|3_{\dot c}] = -u_{1a}\tilde u_{3\dot c} \tag{4.6a}$$

$$\langle 2_b|3_{\dot c}] = u_{2b}\tilde u_{3\dot c} \qquad \langle 2_b|1_{\dot a}] = -u_{2b}\tilde u_{1\dot a} \tag{4.6b}$$

$$\langle 3_c|1_{\dot a}] = u_{3c}\tilde u_{1\dot a} \qquad \langle 3_c|2_{\dot b}] = -u_{3c}\tilde u_{2\dot b} \tag{4.6c}$$

hold. In terms of our choice of spinors, we can choose $u_i = (0, N_i)$ and $\tilde{u}_i = (0, \tilde{N}_i)$; then the solution of the Equations (4.6) can be written as

$$N_2 = \frac{\langle 23\rangle}{\langle 31\rangle}N_1, \quad N_3 = \frac{\langle 23\rangle}{\langle 12\rangle}N_1, \quad \tilde{N}_1 = \frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle}\frac{1}{N_1}, \quad \tilde{N}_2 = \frac{\langle 12\rangle}{N_1}, \quad \tilde{N}_3 = \frac{\langle 31\rangle}{N_1}. \tag{4.7}$$

More general solutions of Equations (4.6) can be obtained by little group transforming this explicit solution. Notice that the overall normalization of these SU(2) spinors u and $\tilde{u}$ is not determined, but that a change in normalization of the $u_i \to N u_i$ has the opposite effect on the $\tilde{u}_i \to 1/N\tilde{u}_i$.

We can establish a key property of the u and $\tilde{u}$ spinors by studying conservation of momentum. In spinorial terms, momentum conservation reads

$$|1^{\dot{a}}][1_{\dot{a}}| + |2^{\dot{b}}][2_{\dot{b}}| + |3^{\dot{c}}][3_{\dot{c}}| = 0 = |1^a\rangle\langle 1_a| + |2^b\rangle\langle 2_b| + |3^c\rangle\langle 3_c|. \tag{4.8}$$

Consider contracting the first half of this statement with $\langle 1_a|$. We find

$$0 = \langle 1_a|2^{\dot{b}}][2_{\dot{b}}| + \langle 1_a|3^{\dot{c}}][3_{\dot{c}}| = u_{1a}\tilde{u}_2^{\dot{b}}[2_{\dot{b}}| - u_{1a}\tilde{u}_3^{\dot{c}}[3_{\dot{c}}|, \tag{4.9}$$

so that $\tilde{u}_2^{\dot{b}}[2_{\dot{b}}| = u_3^{\dot{c}}[3_{\dot{c}}|$. Similarly we find $\tilde{u}_1^{\dot{a}}[1_{\dot{a}}| = u_3^{\dot{c}}[3_{\dot{c}}|$ and that $u^a\langle i_a| = u^a\langle j_a|$ for all i, j. Since we will frequently encounter little group contractions such as $u^a\langle 1_a|$ in the following, we will denote them as $\langle u \cdot 1|$.

As we mentioned earlier in this section, the next ingredient we need is an inverse of each of the SU(2) spinors u_i and $\tilde{u}_i$. We define w_i and $\tilde{w}_i$ so that

$$u_{ia}w_{ib} - u_{ib}w_{ia} = \epsilon_{ab}, \quad \tilde{u}_{i\dot{a}}\tilde{w}_{i\dot{b}} - \tilde{u}_{i\dot{b}}\tilde{w}_{i\dot{a}} = \epsilon_{\dot{a}\dot{b}}. \tag{4.10}$$

for all i. The $w_i, \tilde{w}_i$ are not uniquely specified; given one choice of w_i, for example, then the choice $w_i' = w_i + b_i u_i$ is equally good. We will reduce this b redundancy a little, but we will not fully eliminate it. The additional constraint we impose is motivated by conservation of momentum, which can now be written in the form

$$|1 \cdot u\rangle\left(\langle w_1 \cdot 1| + \langle w_2 \cdot 2| + \langle w_3 \cdot 3|\right) - \left(|w_1 \cdot 1\rangle + |w_2 \cdot 2\rangle + |w_3 \cdot 3\rangle\right)\langle u_1 \cdot 1| = 0. \tag{4.11}$$

We impose the stronger equation

$$|w_1 \cdot 1\rangle + |w_2 \cdot 2\rangle + |w_3 \cdot 3\rangle = 0. \tag{4.12}$$

There is still residual redundancy: we can shift $w_i \to w_i + b_i u_i$ where $b_1 + b_2 + b_3 = 0$. In view of this remaining redundancy, it is interesting to ask what tensors we can construct from the u's and the w's which are invariant under a b change. It is easy to see that the quantities

$$\Gamma_{abc} = u_{1a}u_{2b}w_{3c} + u_{1a}w_{2b}u_{3c} + w_{1a}u_{2b}u_{3c}, \tag{4.13}$$

$$\tilde{\Gamma}_{\dot{a}\dot{b}\dot{c}} = \tilde{u}_{1\dot{a}}\tilde{u}_{2\dot{b}}\tilde{w}_{3\dot{c}} + \tilde{u}_{1\dot{a}}\tilde{w}_{2\dot{b}}\tilde{u}_{3\dot{c}} + \tilde{w}_{1\dot{a}}\tilde{u}_{2\dot{b}}\tilde{u}_{3\dot{c}} \tag{4.14}$$

are invariant; for example, under a b shift, Γ shifts by $\sum_i b_i u_{1a}u_{2b}u_{3c} = 0$. In addition, the quantity $\Gamma\tilde{\Gamma}$ is unambiguously normalized. To gain some more intuition for the physical meaning of these objects, let us examine them in terms of the explicit solution we have obtained for the u's. From the definition of w_i we find that

$$w_{ia} = \begin{pmatrix} \frac{1}{N_i} \\ b_i N_i \end{pmatrix}, \quad (\text{no sum on } i) \tag{4.15}$$

so that eq. (4.12) becomes

$$\begin{pmatrix} -\frac{1}{N_1}\ell_1 - \frac{1}{N_2}\ell_2 - \frac{1}{N_3}\ell_3 \\ b_1 N_1 \tilde{\ell}_1 + b_2 N_2 \tilde{\ell}_2 + b_3 N_3 \tilde{\ell}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.16}$$

Upon substitution of the explicit solutions for the N_i, and use of the equations

$$\tilde{\ell}_2 = \frac{\langle 31\rangle}{\langle 23\rangle}\tilde{\ell}_1, \quad \tilde{\ell}_3 = \frac{\langle 12\rangle}{\langle 23\rangle}\tilde{\ell}_1, \quad \ell_1 + \frac{\langle 31\rangle}{\langle 23\rangle}\ell_2 + \frac{\langle 12\rangle}{\langle 23\rangle}\ell_3 = 0, \tag{4.17}$$

which follow from conservation of momentum in the four dimensional formalism, we see that eq. (4.16) is satisfied when $b_1 + b_2 + b_3 = 0$, as anticipated. As for the quantity $\Gamma\tilde{\Gamma}$, let us content ourselves with an examination of one component. For example, we find that

$$\Gamma_{221}\tilde{\Gamma}_{221} = \frac{N_1 N_2}{N_3}\frac{\tilde{N}_1\tilde{N}_2}{\tilde{N}_3} = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}. \tag{4.18}$$

Notice that this is proportional to $A_3(1^-2^-3^+)$ from four dimensional Yang-Mills theory. Our next task will be to see why this is so.

We begin in familiar territory. The usual color-ordered amplitude in non-Abelian gauge theory is given in terms of polarization vectors by

$$\begin{aligned} A_3 = {} & \frac{i}{\sqrt{2}} \\ & \times\left([\varepsilon_{1a\dot{a}}\cdot\varepsilon_{2b\dot{b}}][\varepsilon_{3c\dot{c}}\cdot(p_1-p_2)]+[\varepsilon_{2b\dot{b}}\cdot\varepsilon_{3c\dot{c}}][\varepsilon_{1a\dot{a}}\cdot(p_2-p_3)]+[\varepsilon_{3c\dot{c}}\cdot\varepsilon_{1a\dot{a}}][\varepsilon_{2b\dot{b}}\cdot(p_3-p_1)]\right). \end{aligned} \tag{4.19}$$

We must rewrite this amplitude in a manifestly gauge-invariant form. A key observation is that

$$\varepsilon_{1a\dot{a}}\cdot p_2 = -\frac{1}{\sqrt{2}}u_{1a}\tilde{u}_{1\dot{a}}. \tag{4.20}$$

Inner products of two polarization vectors are not gauge invariant so we do not expect a simple expression for such inner products. We will choose the same gauge μ, $\tilde{\mu}$ for all three particles. Then, from the spinorial definitions of the polarization vectors, we find that

$$\varepsilon_{1a\dot{a}}\cdot\varepsilon_{2b\dot{b}} = -\langle 1_a|\tilde{\mu}_{\dot{e}}][\tilde{\mu}_{\dot{e}}|2^\beta\rangle^{-1}[2_{\dot{b}}|\mu_d\rangle\langle\mu_d|1^{\dot{a}}]^{-1} \tag{4.21}$$

$$= -\langle 2_b|\tilde{\mu}_{\dot{e}}][\tilde{\mu}_{\dot{e}}|1^\alpha\rangle^{-1}[1_{\dot{a}}|\mu_d\rangle\langle\mu_d|2^{\dot{b}}]^{-1}, \tag{4.22}$$

with similar equations holding for the other inner products. From this point, one systematically uses the definitions of the u's and the w's to remove all of the matrices from the expression for the amplitude. After some work, we find the desired result: the amplitude is

$$A_3(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}) = i\Gamma_{abc}\tilde{\Gamma}_{\dot{a}\dot{b}\dot{c}}. \tag{4.23}$$

We could now continue to express the amplitude in terms of spinor contractions $\langle i|j]$ and their appropriately defined pseudo-inverses; however, we find it to be more convenient not to do this.

4.2 Three point amplitude in gravity

Next, let us consider the three point function in linearized gravity. From the point of view of the SO(4) little group, the graviton polarization tensor is a traceless, symmetric tensor. That is, writing the polarization tensor as ε^{μ}_{mn} where $\mu = 0, \ldots 5$ is a six dimensional Lorentz index and $m, n = 1, \ldots 4$ are vector indices of SO(4), we know that the equations $\varepsilon_{mn} = \varepsilon_{nm}$ and $\sum_m \varepsilon_{mm} = 0$ hold. However, we will find it convenient to also include the antisymmetric tensor and dilaton components, thus enlarging the polarization tensor into an arbitrary two index tensor of SO(4). Contracting this tensor against four dimensional σ matrices, the polarization is $\varepsilon_{a\dot{a};a'\dot{a}'} = \varepsilon_{mn}\sigma^m_{a\dot{a}}\sigma^n_{a'\dot{a}'}$. Note that in this case the helicity of each graviton scattering is labeled by four indices $(a, \dot{a}, a', \dot{a}')$.

At this point we invoke the KLT relations [18], which relate amplitudes in gravity to the square of amplitudes in Yang-Mills theory:

$$\mathcal{M}_3 = A_3 A_3 \tag{4.24}$$

where $\mathcal{M}_3$ is the gravitational three point function. We immediately deduce the simple formula

$$\mathcal{M}_3(1_{aa'\dot{a}\dot{a}'}, 2_{bb'\dot{b}\dot{b}'}, 3_{cc'\dot{c}\dot{c}'}) = -\Gamma_{abc}\Gamma_{a'b'c'}\tilde{\Gamma}_{\dot{a}\dot{b}\dot{c}}\tilde{\Gamma}_{\dot{a}'\dot{b}'\dot{c}'}. \tag{4.25}$$

This equation describes scattering of all possible polarizations of gravitons in six dimensions; reducing to four dimensions, we can deduce expressions for gravitons interacting with gauge bosons and so on.

5 The BCFW recursion relations

With the three point amplitudes in hand, it is straightforward to construct all higher point amplitudes via the BCFW recursion relations. In this section we briefly review these relations and describe an efficient computational method appropriate in dimensions greater than four. We then express the recursion relations in the language of the six dimensional spinor-helicity formalism.

5.1 A review of BCFW

The BCFW recursion relations are an expression for on-shell amplitudes in terms of sums of products of lower point on-shell amplitudes evaluated at complex momenta. First proposed for tree-level YM amplitudes [9, 10], recursion relations were later derived for gravity [12, 13] and eventually found to be a quite generic property of tree amplitudes in quantum field theories in arbitrary dimensions [14, 15]. The basic idea of the recursion relations is to analytically continue two of the external momenta, p_1 and p_2, of an amplitude by a complex parameter, z:

$$\hat{p}_1 = p_1 + zq \tag{5.1}$$

$$\hat{p}_2 = p_2 - zq \tag{5.2}$$

where $q^2 = p_1 \cdot q = p_2 \cdot q = 0$. Note that $\hat{p}_{1,2}$ are complex but still on-shell, and q has the properties of a polarization vector. Since the amplitude A is a rational function of the

JHEP07(2009)075

external momenta, it is an analytic function of z. Consequently, $A(z)$ is defined uniquely by its poles, and the residue at each pole in z corresponds to a product of two on-shell lower point amplitudes evaluated at complex momenta. An appropriate choice of the shift q then lead to the following formula for amplitudes in Yang-Mills theory and gravity:

$$A(p_1,p_2,\ldots)=\sum_{L,R}\sum_{h,h'}\left(\frac{iP_{h,h'}}{k^2}\right)A_L(\hat{p}_1(z_*),\ldots,\hat{k}(z_*);h)A_R(\hat{p}_2(z_*),\ldots,-\hat{k}(z_*);h')\Big|_{z_*=-2q\cdot p_L} \quad (5.3)$$

where $\hat{k}(z_*)$ and k denoted the the shifted momentum and physical momentum of the intermediate leg, respectively, while h and h' labels its polarizations, the ellipsis $\cdots$ denotes the other external momenta, and L,R sums over partitions of the external legs into two groups. The operator $P_{h,h'}$ is the sum over a complete set of propagating states which occurs in the numerator of a propagator.

In order for the recursion relation to hold, it is essential that $A(z)$ not have a pole at $z=\infty$. Thus, any amplitude which satisfies this condition obeys a BCFW-like recursion relation. As argued in [14], $A(z)$ possesses this asymptotic behavior in Yang-Mills theory (gravity) as long as the polarization of particle 1 is $\varepsilon_1^\mu = q^\mu$ ($\varepsilon_1^{\mu\nu} = q^\mu q^\nu$), while the polarization of particle 2 is arbitrary. Since this choice can always be made, BCFW applies to all gauge and gravity theories. Fundamentally, this nice large z behavior occurs because the amplitude must satisfy (complexified) Ward identities, and there is an enhanced "spin Lorentz symmetry" which is present in the theory at large complex momentum transfers. This analysis was later generalized by [15] to include amplitudes in generic theories of diverse spins and particle species, and by [4] to include theories with maximal SUSY. In [20, 21], the authors give a proof of the nice large z behavior of YM theory via complex factorization in four dimensions, independently of any Lagrangian formulation.

5.2 Covariantizing the recursion relation

Ultimately, our goal is to compute a matrix of amplitudes whose matrix elements correspond to each possible choice of the external polarizations. In conventional field theory the nearest approximation to this is the usual amplitude, $A_{\mu_1\mu_2}$, where μ_1 and μ_2 are dotted into the polarization six-vectors for particles 1 and 2. However, as we know, this particular representation is gauge redundant. Instead, we want the object $A_{h_1h_2} = v_{h_1}^{\mu_1} A_{\mu_1\mu_2} v_{h_2}^{\mu_2}$, where v is a basis for the external polarization states labeled by a little group index $h_{1,2}$. For example, in the case of four dimensions this index labels helicity, so $h_{1,2}=\pm$. Consequently, the matrix elements of $A_{h_1h_2}$ correspond to every combination of helicities for particles 1 and 2: A_{--}, A_{-+}, A_{+-} and A_{++}. Likewise, in six dimensions h_1 labels $(a,\dot{a})$ indices, h_2 labels $(b,\dot{b})$ indices, etc.

Unfortunately, conventional BCFW is poorly equipped to evaluate $A_{h_1h_2}$ because it only applies when the deformation vector, q, is chosen to be equal to the polarization of particle 1, $q=\varepsilon_1$. However, this q enters ubiquitously into the right hand side of the BCFW reduction — thus to evaluate $A_{h_1h_2}$ it would be necessary to apply the recursion relations for every linearly independent choice of q! Luckily, there is a simple way around this, which is to choose $q = X_h v_h$ to be an arbitrary linear combination of the v's labeled by a little group vector X; we then use the recursion relations to compute $X_h A_{h,h_2,\ldots}$, the amplitude with appropriate polarization of particle 1. This is the same as the usual BCFW shift except that we are keeping the deformation direction unspecified — as such, the recursion relations do not manifestly break little group covariance. A key point is that the result of the computation $X_h A_{h,h_2,\ldots}$ is linear in X_h; after all, this result is simply the amplitude for particles scattering with particle 1, where particle 1 is in the polarization state $X_h v_h$. Therefore, it is straightforward to deduce the full amplitude $A_{h_1,h_2,\ldots}$ as the coefficient of X_h. We will demonstrate this procedure in examples below. As a final comment, we note that in order to keep $\hat{p}_{1,2}$ on-shell, we must demand that $q^2 = X_h X_{h'}(v_h \cdot v_{h'}) = 0$.

5.3 Application to six dimensions

Thus far, the discussion of the BCFW recursion relations have been independent of space-time dimensionality; in this section, we specialize to six dimensions and introduce some notation that we will use to compute the four and five point amplitudes below. We begin with a simplifying choice of gauge: we take the gauge of particle 1 to be p_2. Then our modified BCFW deformations become

$$\hat{p}_1 = p_1 + zX^{a\dot{a}}\varepsilon_{1a\dot{a}} \tag{5.4}$$

$$\hat{p}_2 = p_2 - zX^{a\dot{a}}\varepsilon_{1a\dot{a}}. \tag{5.5}$$

where the on-shell constraint, $\hat{p}_{1,2}^2 = 0$, fixes $X^{a\dot{a}}X^{b\dot{b}}\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} = 2\det X = 0$. Since X has zero determinant, it is convenient to express it as $X^{a\dot{a}} = x^a\tilde{x}^{\dot{a}}$ and to define

$$y^b = \tilde{x}^{\dot{a}}\langle 2_b|1^{\dot{a}}]^{-1}, \quad \tilde{y}_{\dot{b}} = x^a\langle 1^a|2_{\dot{b}}]^{-1}. \tag{5.6}$$

Then we find that we can implement the vectorial shifts in eq. (5.4) by the spinorial shifts

$$|\hat{1}_a\rangle = |1_a\rangle + zx_a|y\rangle \tag{5.7}$$

$$|\hat{2}_b\rangle = |2_b\rangle + zy_b|x\rangle \tag{5.8}$$

$$|\hat{1}_{\dot{a}}] = |1_{\dot{a}}] - z\tilde{x}_{\dot{a}}|\tilde{y}] \tag{5.9}$$

$$|\hat{2}_{\dot{b}}] = |2_{\dot{b}}] - z\tilde{y}_{\dot{b}}|\tilde{x}], \tag{5.10}$$

where $|x\rangle = x^a|1_a\rangle$ and so on. The BCFW recursion relations for gauge theory can then be written

$$x^a\tilde{x}^{\dot{a}}A_{a\dot{a}b\dot{b}\ldots}(p_1,p_2,\ldots) = \sum_{L,R}\sum_{c\dot{c}}\left(-\frac{i}{k^2}\right)x^a\tilde{x}^{\dot{a}}A_{a\dot{a}c\dot{c}}(\hat{p}_1(z^*),\ldots,\hat{k})A_{b\dot{b}}{}^{c\dot{c}}(\hat{p}_2(z^*),\ldots,-\hat{k}), \tag{5.11}$$

where k and $(c,\dot{c})$ are the momentum and polarization of the intermediate leg. It is worth noticing that a single BCFW computation in six dimensions allows one to deduce results for the scattering of particles in all possible helicity states in four dimensions by a simple dimensional reduction.

JHEP07(2009)075

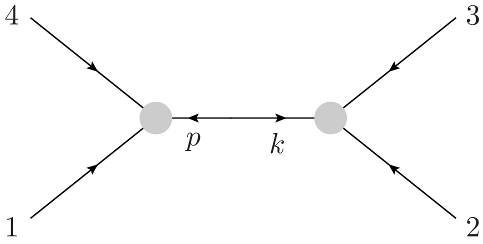

Figure 1. BCFW diagram for the four point amplitude.

6 Computing the four point amplitude

It is now rather easy to use BCFW recursion to compute a compact formula for the four point amplitudes of gauge theory and gravity. We choose to shift the momenta of particles 1 and 2; then there is one BCFW diagram, figure 1. The four point function is given by

$$x^a\tilde{x}^{\dot a} A_{4;a\dot a b\dot b c\dot c d\dot d} = \frac{i}{t} x^a \tilde{x}^{\dot a} A_{L;a\dot a e\dot e d\dot d} A_{R;b\dot b c\dot c}{}^{e\dot e} \tag{6.1}$$

where A_L, A_R are the left- and right-hand three point function in the figure, respectively. Since the three point amplitudes are products of dotted and undotted tensors we can focus our discussion on the undotted tensor. We must compute

$$\Gamma_{L;aed}\Gamma_{R;bc}{}^{e} = (u_{1a}u_{pe}w_{4d}+u_{1a}w_{pe}u_{4d}+w_{1a}u_{pe}u_{4d})(u_{2b}u_{3c}w^e_k+u_{2b}w_{3c}u^e_k+w_{2b}u_{3c}u^e_k). \tag{6.2}$$

It is helpful to choose the spinors associated with the momentum $p=-k$ to be $|p]=i|k]$ and $|p\rangle = i|k\rangle$. Then we find three key properties of the u and w spinors associated with p and k. These are, firstly, that

$$\tilde{u}_p\cdot\tilde{u}_k u_p\cdot u_k = \tilde{u}_p^{\dot e}\tilde{u}_{k\dot e} u_p^e u_{pe} = -s, \tag{6.3}$$

so that, in particular, $u_p\cdot u_k \neq 0$. Consequently, we can exploit the b redundancy of w_k and w_p to choose, secondly,

$$u_p\cdot w_k = \tilde{u}_p\cdot\tilde{w}_k = w_p\cdot u_k = \tilde{w}_p\cdot\tilde{u}_k = 0. \tag{6.4}$$

For, if $u_p\cdot w_k\neq 0$, for example, then we can choose $w_k' = w_k + b_k u_k$ such that $u_p\cdot w_k' = u_p\cdot w_k + b_k u_p\cdot u_k = 0$. This equation always has a solution for b_k since $u_p\cdot u_k\neq 0$. Finally, it is easy to show that

$$w_k\cdot w_p = \frac{1}{u_k\cdot u_p}. \tag{6.5}$$

In view of these three properties, we conclude that we can choose normalizations so that

$$\tilde{w}_k = \frac{\tilde{u}_p}{\sqrt{-s}}, \quad w_k = \frac{u_p}{\sqrt{-s}}. \tag{6.6}$$

The undotted tensorial part of the amplitude now simplifies to

$$\begin{aligned}\Gamma_{L;aed}\Gamma_{R;bc}{}^{e} = \frac{1}{u_k\cdot u_p}\big[&u_{1a}u_{2b}u_{3c}u_{4d} - s\,(u_{1a}u_{2b}w_{3c}w_{4d}\\ &+u_{1a}w_{2b}u_{3c}w_{4d}+w_{1a}u_{2b}w_{3c}u_{4d}+w_{1a}w_{2b}u_{3c}u_{4d})\big].\end{aligned} \tag{6.7}$$

However, it is easy to see that

$$\langle \hat{1}_a \hat{2}_b 3_c 4_d \rangle = \big[u_{1a} u_{2b} u_{3c} u_{4d} \\ -s \left(u_{1a} u_{2b} w_{3c} w_{4d} + u_{1a} w_{2b} u_{3c} w_{4d} + w_{1a} u_{2b} w_{3c} u_{4d} + w_{1a} w_{2b} u_{3c} u_{4d} \right) \big]. \tag{6.8}$$

For example, we compute

$$u_1^a u_2^b \langle \hat{1}_a \hat{2}_b 3_c 4_d \rangle = i \langle u_p \cdot k \; u_k \cdot k \; 3_c 4_d \rangle \tag{6.9}$$

$$= -i u_p \cdot u_k \langle 3_c | \not{k} | 4_d \rangle \tag{6.10}$$

$$= -u_p \cdot u_k \; u_{3c} \tilde{u}_k^e \, [p_e | 4_d \rangle \tag{6.11}$$

$$= -s u_{3c} u_{4d}. \tag{6.12}$$

All other components of $\langle \hat{1}_a \hat{2}_b 3_c 4_d \rangle$ can be projected onto the u_i, w_i basis of the tensor product space in the same way. Thus, we find

$$x^a \tilde{x}^{\dot{a}} A_{4; a\dot{a}b\dot{b}c\dot{c}d\dot{d}} = \frac{-i}{st} x^a \tilde{x}^{\dot{a}} \langle \hat{1}_a \hat{2}_b 3_c 4_d \rangle [\hat{1}_{\dot{a}} \hat{2}_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}] = \frac{-i}{st} x^a \tilde{x}^{\dot{a}} \langle 1_a 2_b 3_c 4_d \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}]. \tag{6.13}$$

Since this final expression is manifestly linear in x and $\tilde{x}$, we can deduce that

$$A_{4; a\dot{a}b\dot{b}c\dot{c}d\dot{d}} = \frac{-i}{st} \langle 1_a 2_b 3_c 4_d \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}]. \tag{6.14}$$

With the expression for the Yang-Mills four point function in hand, it is a trivial matter to deduce the gravitational four point function. The KLT relation in this case is

$$\mathcal{M}_4(1,2,3,4) = -is A_4(1,2,3,4) A_4(1,2,4,3), \tag{6.15}$$

so we immediately deduce that

$$\mathcal{M}_4(1,2,3,4) = \frac{i}{stu} \langle 1_a 2_b 3_c 4_d \rangle \langle 1_{a'} 2_{b'} 3_{c'} 4_{d'} \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}] [1_{\dot{a}'} 2_{\dot{b}'} 3_{\dot{c}'} 4_{\dot{d}'}]. \tag{6.16}$$

The compactness of this explicit expression for the gravitational four point amplitude is an illustration of the power of the spinor-helicity formalism. Of course, this occurs simply because these variable capture physical properties of the single particle state with no redundancy.

7 Five points

The final amplitude we will discuss in this work is the five point amplitudes for Yang-Mills theory. We will compute the five point function using the BCFW recursion relations; then the KLT relations can by used to deduce the gravitational amplitude.

As in our discussion of the four point amplitude, we choose to shift the momenta of particles 1 and 2. There are now two BCFW diagrams, shown in figure 2. Using the identity

$$|k_e\rangle \Gamma_{bc}{}^{e} = -u_c |\hat{2}_b\rangle - u_b |3_c\rangle, \tag{7.1}$$

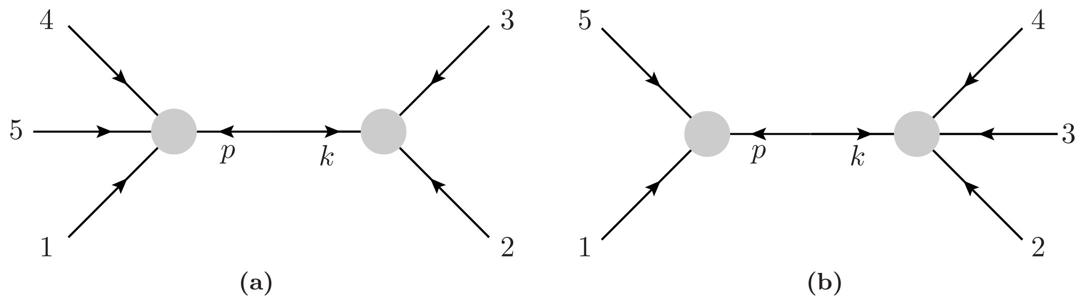

Figure 2. BCFW diagrams for the five point amplitude.

the first diagram can be written as

$$D_1 = \frac{-i}{s_{45}\hat{s}_{51}s_{23}} \left(\langle \hat{1}_a \hat{2}_b 4_d 5_e\rangle u_c + \langle \hat{1}_a 3_c 4_d 5_e\rangle u_b\right)\left([\hat{1}_{\dot{a}}\hat{2}_{\dot{b}}4_{\dot{d}}5_{\dot{e}}]\tilde{u}_{\dot{c}} + [\hat{1}_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\tilde{u}_{\dot{b}}\right), \tag{7.2}$$

where $s_{ij} = (p_i + p_j)^2$ and $\hat{s}_{51} = (\hat{p}_1 + p_5)^2$. Since the first step in these calculations is to express the sum of diagrams (appropriately contracted with $x\tilde{x}$) in a form which is manifestly linear in x and $\tilde{x}$, our first goal in simplifying each diagram is to make the x and $\tilde{x}$ dependence as clear and as simple as possible. In this vein, we define $zq \equiv \hat{p}_1(z) - p_1$, and study $\hat{s}_{15}$. Notice that

$$\hat{s}_{15} q \cdot p_3 = s_{15} q \cdot p_3 + s_{23} q \cdot p_5 = \frac{1}{2s_{12}} \langle x|\not{p}_5 \not{p}_4 \not{p}_3 \not{p}_2|\tilde{x}] \equiv x^a \tilde{x}^{\dot{a}} \phi_{a\dot{a}}. \tag{7.3}$$

Furthermore, we find that

$$q \cdot p_3 = \frac{1}{2s_{12}} \langle x|2^{\dot{b}}]\tilde{u}_{2\dot{b}} u_{2b}[\tilde{x}|2^b\rangle. \tag{7.4}$$

Putting these results together, and contracting in with $x^a \tilde{x}^{\dot{a}}$, we find

$$\begin{aligned} X \cdot D_1 \, = \, & \frac{i}{2(x \cdot \phi \cdot \tilde{x}) s_{12} s_{23} s_{45}} \Big(-\langle x 2_b 4_d 5_e\rangle \langle x|\not{p}_2|3_c\rangle + \langle x 3_c 4_d 5_e\rangle \langle x|\not{p}_3|2_b\rangle \Big) \\ & \times \Big(-[\tilde{x} 2_{\dot{b}} 4_{\dot{d}} 5_{\dot{e}}][\tilde{x}|\not{p}_2|3_{\dot{c}}] + [\tilde{x} 3_{\dot{c}} 4_{\dot{d}} 5_{\dot{e}}][\tilde{x}|\not{p}_3|2_{\dot{b}}] \Big) . \end{aligned} \tag{7.5}$$

Similarly, we find for the second diagram in the figure,

$$\begin{aligned} X \cdot D_2 = & \frac{i}{2(x \cdot \phi \cdot \tilde{x}) s_{12}^2 s_{34} s_{15}} \Big(\langle 2_b 3_c 4_d 5_e\rangle \langle x|\not{p}_5 \not{p}_2|\tilde{x}] - s_{12}\langle 2_b 3_c 4_d x\rangle \langle 5_e|\tilde{x}] + s_{15}\langle 2_b|\tilde{x}]\langle x 3_c 4_d 5_e\rangle \Big) \\ & \times \Big([2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}} 5_{\dot{e}}][\tilde{x}|\not{p}_5 \not{p}_2|x\rangle - s_{12}[2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}} \tilde{x}][5_{\dot{e}}|\tilde{x}\rangle + s_{15}[2_{\dot{b}}|\tilde{x}\rangle[\tilde{x} 3_{\dot{c}} 4_{\dot{d}} 5_{\dot{e}}] \Big) . \end{aligned} \tag{7.6}$$

Notice that neither of the diagrams is linear in x or $\tilde{x}$. However, their sum is linear as expected. This is most easily seen by using the Schouten identity, eq. (A.8) to rewrite the first diagram in the form

$$\begin{aligned} X \cdot D_1 \, = \, & \frac{i}{2(x \cdot \phi \cdot \tilde{x}) s_{12} s_{23} s_{45}} \Big(\langle x 2_b 3_c 4_d\rangle \langle x|\not{p}_4|5_e\rangle - \langle x 2_b 3_c 5_e\rangle \langle x|\not{p}_5|4_d\rangle \Big) \\ & \times \Big([\tilde{x} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}][\tilde{x}|\not{p}_4|5_{\dot{e}}] - [\tilde{x} 2_{\dot{b}} 3_{\dot{c}} 5_{\dot{e}}][\tilde{x}|\not{p}_5|4_{\dot{d}}] \Big) . \end{aligned} \tag{7.7}$$

Meanwhile, another use of the Schouten identity allows us to remove some of the x dependence in the denominator of the second diagram. In particular, we can write the dotted tensor structure, for example, of diagram 2 as

$$\begin{aligned}&\frac{1}{s_{12}}\Big(\langle 2_b3_c4_d5_e\rangle\langle x|\not{p}_5\not{p}_2|\tilde{x}] - s_{12}\langle 2_b3_c4_dx\rangle\langle 5_e|\tilde{x}] + s_{15}\langle 2_b|\tilde{x}]\langle x3_c4_d5_e\rangle\Big)\\ &\quad = -\frac{2(x\cdot\phi\cdot\tilde{x})}{s_{23}}\langle 2_b3_c4_d5_e\rangle + \langle 2_b3_c4_dx\rangle\frac{[\tilde{x}|\not{p}_2\not{p}_3\not{p}_4\not{p}_1|5_e\rangle}{s_{12}s_{23}} + \langle 5_ex2_b3_c\rangle\frac{[\tilde{x}|\not{p}_5\not{p}_1\not{p}_2\not{p}_3|4_d\rangle}{s_{12}s_{23}}. \qquad (7.8)\end{aligned}$$

To complete the cancellation of the quantity $(x\cdot\phi\cdot\tilde{x})$, we simply use the rearrangement formulae given in appendix B. Some further use of the Schouten identity then yields an expression for the five point function which is most conveniently described in terms of two tensors:

$$A_{5;a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} = \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}}\left(\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} + \mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}}\right), \qquad (7.9)$$

where

$$\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} = \langle 1_\alpha|\not{p}_2\not{p}_3\not{p}_4\not{p}_5|1_{\dot{a}}]\langle 2_b3_c4_d5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \text{ cyclic permutations}, \qquad (7.10)$$

and

$$\begin{aligned}2\mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} =\ & \langle 1_a(2.\tilde{\Delta}_2)_{\dot{b}}]\langle 2_b3_c4_d5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \langle 3_c(4.\tilde{\Delta}_4)_{\dot{d}}]\langle 1_a2_b4_d5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}5_{\dot{e}}]\\ &+\langle 4_d(5.\tilde{\Delta}_5)_{\dot{e}}]\langle 1_a2_b3_c5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] - \langle 3_c(5.\tilde{\Delta}_5)_{\dot{e}}]\langle 1_a2_b4_d5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}]\\ &-\big([1_{\dot{a}}(2.\Delta_2)_b\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\langle 1_a3_c4_d5_e\rangle + [3_{\dot{c}}(4.\Delta_4)_d\rangle[1_{\dot{a}}2_{\dot{b}}4_{\dot{d}}5_{\dot{e}}]\langle 1_a2_b3_c5_e\rangle\\ &+[4_d(5.\Delta_5)_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}5_{\dot{e}}]\langle 1_a2_b3_c4_d\rangle - [3_c(5.\Delta_5)_e\rangle[1_{\dot{a}}2_{\dot{b}}4_{\dot{d}}5_{\dot{e}}]\langle 1_a2_b3_c4_d\rangle\big), \qquad (7.11)\end{aligned}$$

where the matrices Δ_i are defined by

$$\Delta_1 = \langle 1|\not{p}_2\not{p}_3\not{p}_4 - \not{p}_4\not{p}_3\not{p}_2|1\rangle, \quad \tilde{\Delta}_1 = [1|\not{p}_2\not{p}_3\not{p}_4 - \not{p}_4\not{p}_3\not{p}_2|1], \qquad (7.12)$$

with the other Δ_i defined by cyclic permutations of this formula. Notice that, while the tensor $\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}}$ is manifestly symmetric under cyclic permutations of the particle label, $\mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}}$ does not obviously have this symmetry. However, it is easy to see that it is symmetric using the Schouten identity.

The gravitational amplitude can then be obtained using the KLT relation. In the case of a five point amplitude, this relation is

$$\mathcal{M}_5 = s_{23}s_{45}A(1,2,3,4,5)A(1,3,2,5,4) + s_{24}s_{35}A(1,2,4,3,5)A(1,4,2,5,3). \qquad (7.13)$$

It is now a matter of algebra to deduce an expression for the gravitational five point amplitude.

8 Concluding remarks

The main achievement of this work has been to find a viable spinor-helicity formalism in six dimensions. That this formalism has the potential to be useful is clear from the simplicity of amplitudes in this framework. In particular, it is remarkable that we now have an explicit, gauge independent, compact formula for the gravitational four point amplitude, given in eq. (6.16). Since it has been possible to extend the spinor-helicity formalism from four to six dimensions, it is fair to ask whether the same is possible for even higher dimensions. A formalism in ten dimensions, for example, might be particularly interesting from the point of view of Yang-Mills theory.

We believe that scattering amplitudes take a remarkable simple form in terms of spinors because these variables encode precisely the physical degrees of freedom of asymptotic states. In particular, amplitudes expressed as functions of spinors transform appropriately under the little group without the need for an unphysical gauge redundancy. That is, the success of the formalism has a physical motivation — it is not a mathematical trick.

In the arena of six dimensions, there are many interesting questions that presently are unanswered. Parke and Taylor [16] wrote down a compact formula for n-point MHV scattering amplitudes in four dimensions. This class of n-point amplitudes is particularly simple, so one might want to examine a simplified subset of amplitudes in six dimensions. However, it is impossible to find such a subset which is closed under Lorentz transformations, because all of the polarization states in six dimensions are connected by a continuous SO(4) symmetry and there is no conserved helicity. The flip side of this statement is that any expression for the n point amplitudes in six dimensions would amount to complete knowledge of the tree-level S-matrix - such an expression would be an exciting discovery.

The BCFW recursion relations apparently allow us to compute amplitudes beginning with the three point amplitude. These three point amplitudes are highly constrained by little group transformation properties, so one can ask the ask whether it is possible to compute amplitudes without recourse to any Lagrangian description of the theory. Recently, an obstacle to this kind of program has been resolved in four dimensions: the work of [20, 21] has shown that the large z behaviour of BCFW-deformed tree amplitudes can be derived on the basis of complex factorization without any use of a Lagrangian; this information is enough to establish the recursion relations. It would be interesting to understand this point in other dimensions; possibly the spinor-helicity formalism we have developed would allow some progress in this direction in six dimensions.

Finally, while we have presented results for gravity and gauge theory, there are other theories in six dimensions which we have left untouched. One particularly interesting theory is the $(2,0)$ theory [22], about which rather little is known. Perhaps insight into this theory might be obtained using these novel kinematic variables. It would also, of course, be interesting to investigate supersymmetric theories.

A The Clifford algebra

Let us start at the beginning. We work with the mostly negative metric, and define Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

The Clifford algebra is

$$\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = 2\eta^{\mu\nu}. \tag{A.2}$$

We will work with a particular basis of this algebra. The Lorentz group SO(6) is isomorphic to SU(4); the spinors of SO(6) are the fundamentals of SU(4). The antisymmetric tensor of SU(4) is the fundamental of SO(6). Therefore, we can choose a basis of the Clifford algebra so that σ, $\tilde{\sigma}$ are antisymmetric. At the same time, it is convenient to work with a basis which is simply related to a standard choice of γ matrices in four dimensions. Our choice is

$$\sigma^0 = i\sigma_1 \otimes \sigma_2 \qquad \tilde{\sigma}^0 = -i\sigma_1 \otimes \sigma_2 \tag{A.3a}$$
$$\sigma^1 = i\sigma_2 \otimes \sigma_3 \qquad \tilde{\sigma}^1 = i\sigma_2 \otimes \sigma_3 \tag{A.3b}$$
$$\sigma^2 = -\sigma_2 \otimes \sigma_0 \qquad \tilde{\sigma}^2 = \sigma_2 \otimes \sigma_0 \tag{A.3c}$$
$$\sigma^3 = -i\sigma_2 \otimes \sigma_1 \qquad \tilde{\sigma}^3 = -i\sigma_2 \otimes \sigma_1 \tag{A.3d}$$
$$\sigma^4 = -\sigma_3 \otimes \sigma_2 \qquad \tilde{\sigma}^4 = \sigma_3 \otimes \sigma_2 \tag{A.3e}$$
$$\sigma^5 = i\sigma_0 \otimes \sigma_2 \qquad \tilde{\sigma}^5 = i\sigma_0 \otimes \sigma_2. \tag{A.3f}$$

We adopt the convention that the six dimensional σ^μ have lower indices while the $\tilde{\sigma}^\mu$ have upper indices. These objects enjoy the properties

$$\sigma^\mu_{AB}\sigma_{\mu CD} = -2\epsilon_{ABCD}, \tag{A.4}$$
$$\tilde{\sigma}^{\mu AB}\tilde{\sigma}^{CD}_\mu = -2\epsilon^{ABCD}, \tag{A.5}$$
$$\sigma^\mu_{AB}\tilde{\sigma}^{CD}_\mu = -2\left(\delta^C_A\delta^D_B - \delta^D_A\delta^C_B\right), \tag{A.6}$$
$$\mathrm{tr}\,\sigma^\mu\tilde{\sigma}^\nu = 4\eta^{\mu\nu}, \tag{A.7}$$

where $\epsilon_{1234} = \epsilon^{1234} = 1$.

The final identity we will discuss in this appendix is the six-dimensional generalization of the Schouten identity. Since the spinors live in a six dimensional space, linear dependence of five (chiral) spinors implies

$$\langle 1234\rangle\langle 5| + \langle 2345\rangle\langle 1| + \langle 3451\rangle\langle 2| + \langle 4512\rangle\langle 3| + \langle 5123\rangle\langle 4| = 0. \tag{A.8}$$

Of course, a similar equation holds for anti-chiral spinors.

B Rearrangement formulae

These rearrangement formulae are useful for simplifying the sum of the two diagrams encountered in the computation of the 5 point amplitude described in section 7. In the

JHEP07(2009)075

notation we used in our discussion of the five point amplitude, the following identities hold:

$$\begin{aligned}
&\frac{1}{s_{12}^2 s_{23}^2 s_{34} s_{15}} [\tilde{x}|\not{p}_2\not{p}_3\not{p}_4\not{p}_1|5_e\rangle\langle x|\not{p}_2\not{p}_3\not{p}_4\not{p}_1|5_{\dot{e}}] + \frac{1}{s_{12}s_{23}s_{45}}\langle x|\not{p}_4|5_e\rangle[\tilde{x}|\not{p}_4|5_{\dot{e}}] \\
&\qquad = 2\frac{x\cdot\phi\cdot\tilde{x}}{s_{12}s_{23}^2 s_{34}s_{45}s_{51}}\langle 5_e|\not{p}_1\not{p}_4\not{p}_3\not{p}_2\not{p}_1\not{p}_4|5_{\dot{e}}], \qquad \text{(B.1)} \\
&\frac{1}{s_{12}^2 s_{23}^2 s_{34} s_{15}} [\tilde{x}|\not{p}_5\not{p}_1\not{p}_2\not{p}_3|4_d\rangle\langle x|\not{p}_5\not{p}_1\not{p}_2\not{p}_3|4_{\dot{d}}] + \frac{1}{s_{12}s_{23}s_{45}}\langle x|\not{p}_5|4_d\rangle[\tilde{x}|\not{p}_5|4_{\dot{d}}] \\
&\qquad = 2\frac{x\cdot\phi\cdot\tilde{x}}{s_{12}s_{23}^2 s_{34}s_{45}}\langle 4_d|\not{p}_5\not{p}_1\not{p}_2\not{p}_3|4_{\dot{d}}], \qquad \text{(B.2)} \\
&\frac{1}{s_{12}^2 s_{23}^2 s_{34} s_{15}} [\tilde{x}|\not{p}_5\not{p}_1\not{p}_2\not{p}_3|4_d\rangle\langle x|\not{p}_2\not{p}_3\not{p}_4\not{p}_1|5_{\dot{e}}] - \frac{1}{s_{12}s_{23}s_{45}}\langle x|\not{p}_5|4_d\rangle[\tilde{x}|\not{p}_4|5_{\dot{e}}] \\
&\qquad = 2\frac{x\cdot\phi\cdot\tilde{x}}{s_{12}s_{23}^2 s_{34}s_{45}s_{51}}\langle 4_d|\not{p}_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4\not{p}_1|5_{\dot{e}}]. \qquad \text{(B.3)}
\end{aligned}$$

Acknowledgments

DOC is the Martin A. and Helen Chooljian Member at the Institute for Advanced Study, and was supported in part by the US Department of Energy under contract DE-FG02-90ER-40542. We thank Nima Arkani-Hamed, Henriette Elvang, Gil Paz, Jared Kaplan, Michele Papucci and Brian Wecht for useful discussions.

References

[1] E.P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, *Annals Math.* **40** (1939) 149 [*Nucl. Phys. Proc. Suppl.* **6** (1989) 9] [SPIRES].

[2] S. Weinberg, *On unitary representations of the inhomogeneous Lorentz group*, in *Quantum theory of fields: volume I*, Cambridge University Press, Cambridge U.K. (1995).

[3] P. Benincasa and F. Cachazo, *Consistency conditions on the S-matrix of massless particles*, arXiv:0705.4305 [SPIRES].

[4] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the simplest quantum field theory?*, arXiv:0808.1446 [SPIRES].

[5] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359 [SPIRES].

[6] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171] [SPIRES].

[7] T. Kugo and P.K. Townsend, *Supersymmetry and the division algebras*, *Nucl. Phys.* **B 221** (1983) 357 [SPIRES].

[8] R.K. Ellis, W.T. Giele, Z. Kunszt and K. Melnikov, *Masses, fermions and generalized D-dimensional unitarity*, arXiv:0806.3467 [SPIRES].

[9] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308] [SPIRES].

[10] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052] [SPIRES].

[11] S.D. Badger, E.W.N. Glover, V.V. Khoze and P. Svrček, *Recursion relations for gauge theory amplitudes with massive particles*, *JHEP* **07** (2005) 025 [hep-th/0504159] [SPIRES].

[12] F. Cachazo and P. Svrček, *Tree level recursion relations in general relativity*, hep-th/0502160 [SPIRES].

[13] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A recursion relation for gravity amplitudes*, *Nucl. Phys.* **B 721** (2005) 98 [hep-th/0502146] [SPIRES].

[14] N. Arkani-Hamed and J. Kaplan, *On tree amplitudes in gauge theory and gravity*, *JHEP* **04** (2008) 076 [arXiv:0801.2385] [SPIRES].

[15] C. Cheung, *On-shell recursion relations for generic theories*, arXiv:0808.0504 [SPIRES].

[16] S.J. Parke and T.R. Taylor, *An amplitude for n gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459 [SPIRES].

[17] C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *On-shell unitarity bootstrap for QCD amplitudes*, *Nucl. Phys. Proc. Suppl.* **160** (2006) 261 [hep-ph/0610089] [SPIRES].

[18] H. Kawai, D.C. Lewellen and S.H. Henry Tye, *A relation between tree amplitudes of closed and open strings*, *Nucl. Phys.* **B 269** (1986) 1 [SPIRES].

[19] Z. Bern, L.J. Dixon, M. Perelstein and J.S. Rozowsky, *Multi-leg one-loop gravity amplitudes from gauge theory*, *Nucl. Phys.* **B 546** (1999) 423 [hep-th/9811140] [SPIRES].

[20] P.C. Schuster and N. Toro, *Constructing the tree-level Yang-Mills S-matrix using complex factorization*, *JHEP* **06** (2009) 079 [arXiv:0811.3207] [SPIRES].

[21] S. He and H.-B. Zhang, *Consistency conditions on S-matrix of spin* 1 *massless particles*, arXiv:0811.3210 [SPIRES].

[22] E. Witten, *Some comments on string dynamics*, hep-th/9507121 [SPIRES].